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Graphs and Combinatorics

Research Problems

In this column *Graphs and Combinatorics* publishes current research problems whose proposer believes them to be within reach of existing methods.

Manuscripts should preferably contain the background of the problem and all references known to the author. The length of the manuscript should not exceed two type-written pages. Manuscripts should be sent to:

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Hypergraphs with High Chromatic Number

It is easy and well known that every graph $G = (V, E)$ with chromatic number s contains at least as many edges as the complete graph on s vertices. Indeed, if we properly color G by s colors then there is at least one edge joining a vertex of color i to a vertex of color j, for all $1 \le i < j \le s$, since otherwise G can be properly colored by $s - 1$ colors.

Let $f(k, s)$ denote the minimal number of edges of a k-uniform hypergraph whose chromatic number is at least s. By the preceding paragraph

$$f(2, s) = \binom{s}{2} \text{ for all } s \ge 1. \tag{1}$$

The determination of $f(k, s)$ for $k > 2$ seems much more complicated. Clearly, the chromatic number of the complete k-uniform hypergraph on $(s - 1)(k - 1) + 1$ vertices is s and hence

$$f(k, s) \le \binom{(s-1)(k-1)+1}{k} \text{ for all } k \ge 2, s \ge 1. \tag{2}$$

For $k = 2$ inequality (2) is sharp, by (1). It is not, however, sharp for $k > 2$. Thus, for example, one can show that $f(3, 3) = 7$ (the 7 lines of the Fano-plane form a 3 uniform hypergraph whose chromatic number is 3), whereas (2) only gives $f(3, 3) \le 10$.

There are many results about the asymptotic behaviour of $f(k, 3)$ as k tends to infinity (see [1, 2, 5]). The asymptotic behaviour of $f(k, s)$ for fixed k as s tends to infinity is not so well studied. Erdös and Hajnal ([4], see also [3]) conjectured more than twenty years ago that for every fixed k, if $s > s_0(k)$ then equality holds in (2). This conjecture is false, as easily follows from the following observation.

Proposition 1. *For every k, s*

$$f(k,s) \le \binom{(s-1)k+1}{k} \cdot \frac{\log k}{\log k - 1} \cdot \frac{1}{\left\lceil \dfrac{k}{\log k} \right\rceil} \equiv g(k,s).$$

Proof. Recall that the Turán number $T(n,r,t)$ is the minimum cardinality of a family T of t-sets of an n element set X such that each r-element subset of X contains a member of T. By a result of Frankl and Rödl [6] (see also [7] for a previous but slightly weaker result) for every n, $k > 0$

$$T(n, k+1, k) < \frac{\log k}{\log k - 1} \cdot \frac{1}{\lceil k/\log k \rceil} \binom{n}{k}.$$

Therefore, there exists a family E of $g(k,s)$ k-subsets of $V = \{1, 2, \ldots, (s-1)\cdot k + 1\}$ such that every $(k+1)$-subset of V contains a member of E. The chromatic number of $G = (V, E)$ is clearly at least s, since in every $(s-1)$-coloring of it there are $k+1$ vertices having the same color and hence there is a monochromatic edge. □

By inequality (2) for every fixed k $f(k,s) = O(s^k)$ as $s \to \infty$. On the other hand it is not too difficult to show that for every fixed k $f(k,s) = \Omega(s^k)$ as $s \to \infty$. This follows from the following statement.

Proposition 2. *For every k, s*

$$f(k,s) > (k-1)\left\lceil \frac{s-1}{k} \right\rceil \cdot \left\lceil \frac{k-1}{k}(s-1) \right\rceil^{k-1} \equiv h(k,s). \tag{3}$$

Proof. Let $G = (V, E)$ be a k uniform hypergraph with at most $h(k,s)$ edges. We must show that G is $(s-1)$-colorable. Color V randomly with $\left\lceil \frac{k-1}{k}(s-1) \right\rceil$ colors. The expected number of monochromatic edges of G is clearly $|E|/\left\lceil \frac{k-1}{k}(s-1) \right\rceil^{k-1} \le (k-1)\cdot\left\lceil \frac{s-1}{k} \right\rceil$, and thus there is such a coloring with at most that many monochromatic edges. Pick a set of at most $\left\lceil \frac{s-1}{k} \right\rceil \cdot (k-1)$ vertices which intersects each of these edges. Recolor these vertices with the remaining $\lceil (s-1)/k \rceil$ colors, such that each color appears at most $k-1$ times. One can easily check that this forms a proper $(s-1)$-coloring of G. □

By the proof of Proposition 1

$$f(k,s) \le \min_{a \ge 0} T((s-1)(k+a-1)+1, k+a, k). \tag{4}$$

In [6] it is shown that

$$T(n, k+a, k) \le \left(1 - \frac{1}{2k^a}\right) \cdot a \cdot (a+4) \cdot \frac{\log k}{\binom{k}{a}} \cdot \binom{n}{k}.$$

Combining these two inequalities $\left(\text{with } a = \left[\frac{k}{2}\right]\right)$ we can get the following strengthening of Proposition 1 for large values of k.

Proposition 3. *If* $k \to \infty$ *and* $\frac{s}{k} \to \infty$ *then*

$$f(k,s) = O\left(k^{5/2} \cdot \log k \cdot \left(\frac{3}{4}\right)^k \cdot \binom{(s-1)(k-1)+1}{k}\right). \tag{5}$$

Thus, for large k, $f(k,s)$ is quite far from the right hand side of (2) even for $s \gg k$.

It would be interesting to determine or estimate $f(k,s)$ for all k, s and in particular to decide which of the bounds (3) and (5) is closer to $f(k,s)$ for fixed (large) k as s tends to infinity. The following conjecture seems plausible.

Conjecture. For every fixed k the limit

$$\lim_{s\to\infty} f(k,s)/s^k \text{ exists.}$$

It would be interesting to find the value of this limit if it exists, and also to decide how good an estimate for $f(k,s)$ is (4). (One can easily check that (4) is strict for $s = k = 3$.)

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